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TOPOLOGICAL ENTROPY OF EXPANSIVE FLOWS ON UNIFORM SPACES

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Abstract. We define topological entropy of flows on uniform spaces and discuss some properties of topological entropy of expansive flows on uniform spaces.

1. Introduction

We define the topological entropy of dynamical systems on metric spaces using metric. Uniform spaces are generalization of metric spaces. We define the topological entropy of flows on uniform spaces and discuss some properties of topological entropy of expansive flows on uniform spaces.

Let X be a set. The set $\triangle_X = \{(x,x) \mid x \in X\}$ denotes the diagonal in $X \times X$. The inverse α^{-1} of a set $\alpha \subset X \times X$ is defined by $\alpha^{-1} = \{(y,x) \mid (x,y) \in \alpha\}$. If $\alpha^{-1} = \alpha$ then α is called symmetric. The composite $\beta \circ \alpha$ of two subsets α and β of $X \times X$ is defined by

$$\beta \circ \alpha = \{(x,z) \in X \times X \mid (x,y) \in \alpha \text{ and } (y,z) \in \beta \text{ for some } y \in X\}.$$

Note that $\alpha^1 = \alpha, \alpha^{n+1} = \alpha \circ \alpha^n$ and if $\triangle_X \subset \alpha$ then $\alpha \subset \alpha^2 \subset \alpha^3 \subset \cdots$.

Definition 1.1. A uniformity on a set X is a collection $\mathcal{U}$ of subsets of $X \times X$ satisfying the following properties:

(1) each member of $\mathcal{U}$ contains the diagonal $\triangle_X$,
(2) if $\alpha \in \mathcal{U}$ and $\alpha \subset \beta \subset X \times X$, then $\beta \in \mathcal{U}$,
(3) if α and β are members of $\mathcal{U}$, then $\alpha \cap \beta \in \mathcal{U}$,
(4) if $\alpha \in \mathcal{U}$, then $\alpha^{-1} \in \mathcal{U}$,
(5) for every $\alpha \in \mathcal{U}$ there is $\beta \in \mathcal{U}$ such that $\beta^2 \subset \alpha$.

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The set X equipped with a uniformity $\mathcal{U}$ is called a uniform space and an element of $\mathcal{U}$ is called an entourage of X.

Let $x \in X, A \subset X$ and $\alpha \subset X \times X$ we define the following notations

$$\alpha[x] = \{y \in X \mid (x, y) \in \alpha\}$$

and

$$\alpha[A] = \bigcup_{x \in A} \alpha[x].$$

It is obvious that $y \in \alpha[x]$ if and only if $x \in \alpha^{-1}[y]$ and $(\beta \circ \alpha)[x] = \beta[\alpha[x]]$. It is well known that if $(X,\mathcal{U})$ is a uniform space, then the topology $\mathcal{T}$ of the uniformity $\mathcal{U}$ is the collection of all subsets A of X such that for each $x \in A$ there is $\alpha \in \mathcal{U}$ such that $\alpha[x] \subset A$. $\mathcal{T}$ is called the uniform topology generated by $\mathcal{U}$.

Given two uniform spaces X and Y, a map $f : X \to Y$ is said to be uniformly continuous if for each encourage β of Y there is an entourage α of X such that

$$(f(x), f(y)) \in \beta \text{ for all } (x, y) \in \alpha.$$

Let X be a uniform space with uniformity $\mathcal{U}$. A flow on X is a continuous map $\phi : X \times \mathbb{R} \to X$ satisfying

(1) $\phi(x, 0) = x$ for all $x \in X$,
(2) $\phi(x, s + t) = \phi(\phi(x, s), t)$ for all $x \in X, s, t \in \mathbb{R}$.

For $t \in \mathbb{R}$ let ϕ_t be a homeomorphism of X defined by $\phi_t(x) = \phi(x, t)$ for all $x \in X$. Denotes by $C_0(\mathbb{R})$ the set of all continuous functions $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$.

Definition 1.2. A flow $\phi : X \times \mathbb{R} \to X$ is said to be *expansive* if for every $\epsilon > 0$ there is an expansive entourage α such that if $x, y \in X$ satisfy $(\phi_t(x), \phi_{h(t)}(y)) \in \alpha$ for every $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$, then $y = \phi_\tau(x)$ where $|\tau| < \epsilon$.

Theorem 1.3. *Let X be a compact uniform space with uniformity $\mathcal{U}$. The followings are equivalent for a flow ϕ on X without fixed points.*

(1) *ϕ is expansive.*
(2) *For every $\epsilon > 0$ there exists $\alpha \in \mathcal{U}$ such that if $x, y \in X$ satisfty*

$$(\phi_t(x), \phi_{h(t)}(y)) \in \alpha$$

for every $t \in \mathbb{R}$ and some increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$, then $y = \phi_\tau(x)$ for some $\tau \in (-\epsilon, \epsilon)$.

(3) *For any* $\alpha \in \mathcal{U}$ *there is* $\beta \in \mathcal{U}$ *such that if* $x, y \in X$ *satisfy* $(\phi_t(x), \phi_{h(t)}(y)) \in \beta$ *for every* $t \in \mathbb{R}$ *and some* $h \in C_0(\mathbb{R})$*, then* y *is on the same orbit as* x *and the orbit from* x *to* y *lies inside* $\alpha[x]$.

(4) *For any* $\epsilon > 0$ *there exist* $\beta \in \mathcal{U}$ *and* $\tau > 0$ *such that if* $\{t_i\}_{i=-\infty}^{\infty}$, $\{s_i\}_{i=-\infty}^{\infty}$ *are bisequences of real numbers with* $t_0 = s_0 = 0, 0 < t_{i+1} - t_i \le \tau$, $|s_{i+1} - s_i| \le \tau, t_i \to \infty$ *and* $t_{-i} \to -\infty$ *as* $i \to \infty$ *and if* $x, y \in X$ *satisfy* $(\phi_{t_i}(x), \phi_{s_i}(y)) \in \beta$ *for every* $i \in \mathbb{Z}$*, then* $y = \phi_t(x)$ *for some* $t \in (-\epsilon, \epsilon)$.

Proof. See the [3]. □

2. Main theorem

We will define the topological entropy of homeomorphisms of uniform spaces. Let $(X,\mathcal{U})$ be a uniform space and f a homeomorphism of X and let $n \in \mathbb{N}, \alpha \in \mathcal{U}$. For $E, F \subset X$ we say that E (n,α)-spans F with respect to f if for each $x \in F$ there is $y \in E$ such that $(f^k(x), f^k(y)) \in \alpha$ for all $0 \le k < n$. We let $r_n(F, \alpha, f)$ denote the minimum cardinality of a set which (n,α)-spans F with respect to f. If F is compact, then the continuity of f guarantees $r_n(F, \alpha, f) < \infty$. For a compact set $F \subset X$ we define

$$\overline{r}_f(F, \alpha) = \limsup \frac{1}{n} \log r_n(F, \alpha, f)$$

and

$$h(f, F) = \sup \overline{r}_f(F, \alpha).$$

Note that $\overline{r}_f(F, \alpha)$ increases as α decreses.

For $\alpha \in \mathcal{U}$ and $x \in X$ define

$$\Gamma_\alpha(x, f) = \{y \in X \mid (f^n(x), f^n(y)) \in \alpha \text{ for all } n \in \mathbb{Z}\}.$$

f is called h-expansive if there exists $\alpha \in \mathcal{U}$ such that $h(f, \Gamma_\alpha(x, f)) = 0$ for all $x \in X$. A flow ϕ on X is called h-expansive if ϕ_t is h-expansive for all $t > 0$.

THEOREM 2.1. *Let* $(X,\mathcal{U})$ *be a compact uniform space. Every expansive flow* ϕ *on* X *is* h*-expansive.*

Proof. Let $t > 0$. For $\alpha \in \mathcal{U}$ define

$$\Gamma_\alpha(x, \phi) = \{y \in X \mid (\phi_s(x), \phi_s(y)) \in \alpha \text{ for all } s \in \mathbb{R}\}.$$

For any $\tau > 0$, since ϕ is expansive there is $\alpha \in \mathcal{U}$ such that

$$\Gamma_\alpha(x, \phi) \subset \phi_{[-\tau,\tau]}(x).$$

By integral continuity theorem, there exists $\beta \in \mathcal{U}$ such that if $(x, y) \in \beta$ then $(\phi_s(x), \phi_s(y)) \in \alpha$ for all $0 \leq s \leq t$. Let $y \in \Gamma_\beta(x, \phi_t)$. Then

$$(\phi_{nt}(x), \phi_{nt}(y)) = (\phi_t^n(x), \phi_t^n(y)) \in \beta$$

for all $n \in \mathbb{Z}$. For any $s \in \mathbb{R}$, there is $n \in \mathbb{Z}$ such that $nt \leq s < (n+1)t$. Since

$$(\phi_{nt}(x), \phi_{nt}(y)) \in \beta \text{ and } 0 \leq s - nt < t,$$

we have

$$(\phi_{s-nt}(\phi_{nt}(x)), \phi_{s-nt}(\phi_{nt}(y))) = (\phi_s(x), \phi_s(y)) \in \alpha.$$

Thus

$$\Gamma_\beta(x, \phi_t) \subset \Gamma_\alpha(x, \phi) \subset \phi_{[-\tau,\tau]}(x).$$

Since $\phi : X \times [0, 1] \to X$ is uniformly continuous, there exist $\gamma \in \mathcal{U}$ and $c \in (0, 1)$ such that if $(p, q) \in \gamma$, $u, v \in [0, 1]$, $|u - v| < c$, then $(\phi_u(x), \phi_v(y)) \in \beta$ for all $x \in X$.

We claim that if $u, v \in \mathbb{R}$ and $|u - v| < c$, then $(\phi_u(x), \phi_v(y)) \in \beta$ for all $x \in X$.

(a) when $u < v$, since $(\phi_u(x), \phi_u(x)) \in \triangle_X \subset \gamma$ and $0 \leq v - u < c$, we have

$$(\phi_u(x), \phi_{v-u}(\phi_u(x))) = (\phi_u(x), \phi_v(x)) \in \beta,$$

(b) when $u \geq v$, since $(\phi_v(x), \phi_v(x)) \in \triangle_X \subset \gamma$ and $0 \leq v - u < c$, we have

$$(\phi_{u-v}(\phi_v(x)), \phi_v(x)) = (\phi_u(x), \phi_v(x)) \in \beta.$$

Since $\{(s - c, s + c) \mid s \in [-\tau, \tau]\}$ is an open cover of $[-\tau, \tau]$ and $[-\tau, \tau]$ is compact, there exist finitely many $s_1, s_2, \ldots, s_m \in [-\tau, \tau]$ such that

$$[-\tau, \tau] \subset \bigcup_{i=1}^{m} (s_i - c, s_i + c).$$

Let $n \in \mathbb{N}$. We claim that $\{\phi_{s_i}(x) \mid i = 1, 2, \ldots, m\}$ (n, β)-spans $\phi_{[-\tau,\tau]}(x)$ with respect to ϕ_t. For any $s \in [-\tau, \tau]$, there is i such that $s \in (s_i - c, s_i + c)$. For $0 \leq k < n$, since

$$|kt + s - (kt + s_i)| = |s - s_i| < c,$$

we have

$$(\phi_t^k(\phi_s(x)), \phi_t^k(\phi_{s_i}(x))) = (\phi_{kt+s}(x), \phi_{kt+s_i}(x)) \in \beta.$$

Thus $\{\phi_{s_i}(x) \mid i = 1, 2, \ldots, m\}$ (n, β)-spans $\phi_{[-\tau,\tau]}(x)$ with respect to ϕ_t. Hence $r_n(\phi_{[-\tau,\tau]}(x), \beta, \phi_t) \leq m$ for all $n \in \mathbb{N}$. Thus

$$\overline{r}_{\phi_t}(\phi_{[-\tau,\tau]}(x), \beta) = 0.$$

Hence

$$h(\phi_t, \Gamma_\beta(x,\phi_t)) \le h(\phi_t, \phi_{[-\tau,\tau]}(x)) = 0.$$

Therefore ϕ is h-expansive. $\square$

We will now look at the entropy of an expansive flow ϕ on a uniform space X with uniformity $\mathcal{U}$. Let $t > 0$ and $\alpha \in \mathcal{U}$. For $E, F \subset X$ we say that E (t,α)-spans F with respect to ϕ if for each $x \in F$ there is $y \in F$ such that $(\phi_s(x), \phi_s(y)) \in \alpha$ for all $0 \le s \le t$. Let $r_t(F,\alpha,\phi)$ denote the minimum cardinality of a set which (t,α)-spans F with respect to ϕ. We claim that if F is compact then $r_t(F,\alpha,\phi) < \infty$. We may assume that α is symmetric. For each $x \in X$, by integral continuity theorem, there exists a neighborhood U_x of x such that if $y \in U_x$ then $(\phi_s(x), \phi_s(y)) \in \alpha$ for all $0 \le s \le t$. $\{U_x \mid x \in F\}$ is an open cover of F. Since F is compact, there exist finitely many $x_1, x_2, \ldots, x_n \in F$ such that $F \subset \bigcup_{i=1}^n U_{x_i}$. For any $x \in F$, there is i such that $x \in U_{x_i}$. Then $(\phi_s(x), \phi_s(x_i)) \in \alpha^{-1} = \alpha$ for all $0 \le s \le t$. Thus $\{x_1, x_2, \ldots, x_n\}$ (t,α)-spans F with respect to ϕ and so $r_t(F,\alpha,\phi) \le n$. We define $\overline{r}_\phi(F,\alpha) = \limsup \frac{1}{t}\log r_t(F,\alpha,\phi)$.

Let $F \subset X$. We say that $E \subset F$ is a (t,α)-separated subset of F with respect to ϕ if for any $x, y \in E$ with $x \ne y$ we have $(\phi_s(x), \phi_s(y)) \notin \alpha$ for some $0 \le s \le t$. Let $s_t(F,\alpha,\phi)$ denote the maximum cardinality of a set which is a (t,α)-separated subset of F. We claim that if F is compact, then $s_t(F,\alpha,\phi) < \infty$. There exists symmetric $\beta \in \mathcal{U}$ such that $\beta^2 \subset \alpha$. For any $x \in X$, by integral continuity theorem, there exists a neighborhood U_x of x such that if $(x,y) \in U_x$ then $(\phi_s(x), \phi_s(y)) \in \beta$ for all $0 \le s \le t$. $\{U_x \mid x \in F\}$ is an open cover of F. Since F is compact, there exist finitely many $x_1, x_2, \ldots, x_n \in F$ such that $F \subset \bigcup_{i=1}^n U_{x_i}$. If $E \subset F$ with $\sharp E \ge n+1$, then there exist $x, y \in E$ and i such that $x, y \in U_{x_i}$. Since

$$(\phi_s(x), \phi_s(x_i)) \in \beta^{-1} = \beta \quad \text{and} \quad (\phi_s(x_i), \phi_s(y)) \in \beta$$

for all $0 \le s \le t$, we have

$$(\phi_s(x), \phi_s(y)) \in \beta^2 \subset \alpha$$

for all $0 \le s \le t$. Thus E is not (t,α)-separated. Hence $s_t(F,\alpha,\phi) \le n$. We define

$$\overline{s}_\phi(F,\alpha) = \limsup \frac{1}{t}\log s_t(F,\alpha,\phi)$$

and topological entropy by

$$h(\phi, F) = \sup \overline{r}_\phi(F,\alpha) = \sup \overline{s}_\phi(F,\alpha).$$

These limits exist and are equal by following proposition.

PROPOSITION 2.2. (1) *$r_t(F,\alpha,\phi) \le s_t(F,\alpha,\phi) \le r_t(F,\beta)$ where $\beta^2 \subset \alpha$ and $\beta^{-1} = \beta$.*

(2) *If $\alpha \subset \beta$, then we have $\overline{r}_\phi(F,\beta) \le \overline{r}_\phi(F,\alpha)$ and $\overline{s}_\phi(F,\beta) \le \overline{s}_\phi(F,\alpha)$.*

Proof. (1) Let E be a maximal (t,α)-separated subset of F. For any $x \in F - E$, since $E \cup \{x\}$ is not a (t,α)-separated subset of F, there exists $y \in E$ such that

$$(\phi_s(x), \phi_s(y)) \in \alpha$$

for all $0 \le s \le t$. Thus E (t,α)-spans F. Hence $r_t(F,\alpha,\phi) \le s_t(F,\alpha,\phi)$. Let E_1 be a (t,α)-separated subset of F and let E_2 (t,β)-span F. For any $x \in E_1 \subset F$, there exists $f(x) \in E_2$ such that $(\phi_s(x), \phi_s(f(x))) \in \beta$ for all $0 \le s \le t$. If $f(x) = f(y)$, then we have

$$(\phi_s(x), \phi_s(f(x))) \in \beta \text{ and } (\phi_s(f(y)), \phi_s(y)) \in \beta^{-1} = \beta$$

for all $0 \le s \le t$. Thus $(\phi_s(x), \phi_s(y)) \in \beta^2 \subset \alpha$ for all $0 \le s \le t$. Since E_1 is (t,α)-separated, we have $x = y$ and so f is injective. Thus $\sharp E_1 \le \sharp E_2$. Hence

$$s_t(F,\alpha,\phi) \le r_t(F,\alpha,\phi).$$

(2) trivial. □

For $t > 0$, let $\upsilon(t)$ denote the number of closed orbits of ϕ with a period $\tau \in [0,t]$ and $\upsilon_c(t)$ the number of closed orbits of ϕ with a period $\tau \in [t-c, t+c]$.

THEOREM 2.3. *Let ϕ be an expansive flow on a compact uniform space X. Then*

$$\limsup \frac{1}{t} \log \upsilon(t) \le h(t) \equiv h(\phi, X).$$

Proof. Let $\epsilon > 0$. By Theorem 1.3 (4), there exist $\alpha \in \mathcal{U}$ and $\tau > 0$ such that if $(t_i), (u_i)$ are bi-sequences with $t_0 = u_0 = 0$, $0 < t_{i+1} - t_i \le 2\tau$, $|u_{i+1} - u_i| \le 2\tau$ for all $i \in \mathbb{Z}$, $t_i \to \infty$ and $t_{-i} \to -\infty$ as $i \to \infty$ and if $(\phi_{t_i}(x), \phi_{u_i}(y)) \in \alpha$ for all $i \in \mathbb{Z}$, then $y = \phi_t(x)$ for some $|t| < \epsilon$. Let $x, y \in X$ be distinct periodic points with periods $a, b \in [t - \frac{\tau}{2}, t + \frac{\tau}{2}]$ respectively. Let $m = \left[\frac{t-\frac{\tau}{2}}{\tau}\right] + 1$ and put

$$t_{pm+q} = pa + q\tau,\ u_{pm+q} = pb + q\tau$$

for $(p,q) \in \mathbb{Z} \times \{0, 1, \dots, m-1\}$. Then $t_0 = u_0 = 0$,

$$t_{pm+q+1} - t_{pm+q} = u_{pm+q+1} - u_{pm+q} = \tau,$$

for $(p, q) \in \mathbb{Z} \times \{0, 1, \ldots, m-2\}$ and,

$$\begin{aligned} 0 < t_{(p+1)m} - t_{pm+m-1} &\leq 2\tau, \\ 0 < u_{p(m+1)} - u_{pm+m-1} &\leq 2\tau, \end{aligned}$$

$$t_i \to \infty \text{ and } t_{-i} \to -\infty \text{ as } i \to \infty.$$

Suppose that x, y are not (t, α)-separated. Then

$$(\phi_s(x), \phi_s(y)) \in \alpha \text{ for all } 0 \leq s \leq t.$$

Since

$$(\phi_{t_{pm+q}}(x), \phi_{u_{pm+q}}(y)) = (\phi_{q\tau}(\phi_{pa}(x)), \phi_{q\tau}(\phi_{pb}(y))) = (\phi_{q\tau}(x), \phi_{q\tau}(y))$$

and

$$0 \leq q\tau \leq (m-1)\tau < t,$$

we have $(\phi_{t_i}(x), \phi_{u_i}(y)) \in \alpha$ for all $i \in \mathbb{Z}$. Thus $y = \phi_t(x)$ for some $|t| < \epsilon$ and so we have a contradiction. Hence x, y are (t, α)-separated. Therefore we have

$$\upsilon_{\frac{\tau}{2}}(t) \leq r_t(\phi, \alpha).$$

Since $\upsilon(t) \leq \sum_{n=1}^{[\frac{t}{\tau}]} \upsilon_{\frac{\tau}{2}}(n\tau)$ and $r_t(\phi, \alpha)$ increases with t, $\upsilon(t) \leq \frac{t}{\tau} r_t(\phi, \alpha)$. Thus we have

$$\limsup \frac{1}{t} \log \upsilon(t) \leq h(\phi, \alpha) \leq h(\phi).$$

This completes the proof. □

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